Control and Planning of 3D Dynamic Walking with Asymptotically Stable Gait Primitives

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***Abstract*—In this paper we present a hierarchical framework that enables motion planning for asymptotically stable 3D bipedal walking in the same way that planning is already possible for ZMP walking. This framework is based on the construction of *asymptotically stable gait primitives* for a class of hybrid dynamical systems with impacts. Each primitive corresponds to an asymptotically stable hybrid limit cycle that admits rules *a priori* for sequential composition with other primitives, reducing a high-dimensional feedback motion planning problem into a low-dimensional discrete tree search. As a constructive example, we develop this planning framework for the 3D compass-gait biped, where each primitive corresponds to walking along an arc of constant curvature for a fixed number of steps. We apply a discrete search algorithm to plan a sequence of these primitives taking the 3D biped stably from start to goal in workspaces with obstacles. We finally show how this framework generalizes to more complex models by planning walking paths for an underactuated five-link biped.**

I. INTRODUCTION

Passive dynamic walking is characterized by phases of instability where the center of mass engages in pendular falling until ground reaction forces redirect this motion into the next step cycle. This interplay between continuous and discrete dynamics results in repetitive motion that is inherently stable from step to step, i.e., perturbations are asymptotically dissipated over a walking sequence. Many real-world robotic systems such as RABBIT [1], ERNIE [2], MABEL [3], Gibbot [4], and Parkourbot [5] exhibit these *asymptotically stable* gaits. These robots rely on ballistic momentum and gravitational energy to drive their unactuated degrees-of-freedom (DOFs), e.g., passive rotation of the support foot [1]–[3], [6], which contributes to their speed and energetic efficiency.

However, asymptotically stable walkers currently lack the same functionality as humanoid robots, e.g., some require downhill slopes for strictly gravity-powered walking, are constrained to the sagittal plane-of-motion, lack directional control authority, and/or lack redundant joints for manipulation. Hybrid nonlinear dynamics make it difficult to prove stability of a given motion–the robot state cannot be checked at each instant against closed-form balance conditions like Zero Moment Point (ZMP, cf. [7]). For this reason it has been difficult to extend asymptotically stable approaches, which rely on numerical analysis [8], into motion planning applications.

The goal of this paper is to enable motion planning for asymptotically stable dynamic walking in the same way that planning is already possible for ZMP-based walking. We will do so by constructing a set of asymptotically stable "motion primitives" with safety guarantees that are amenable to established planning methods based on ZMP motion primitives.

Motion primitives prescribe a library of common actions such as walking and climbing, reducing the high-dimensional kinodynamic planning problem to a discrete sequence of precomputed motions [9], [10]. Instead of using motion primitives that track ZMP-constrained joint trajectories, we will build a set of control systems yielding asymptotically stable walking gaits. For this purpose we adopt the energy-shaping method of controlled reduction [11], which has previously been used to construct 3D gaits for both straight-ahead walking and constant-curvature steering [12], [13].

Gaits capable of steering with mild curvature have similarly been produced using hybrid zero dynamics in [6]. This approach elegantly exploits the fact that local asymptotic stability implies local input-to-state stability: there exist bounds on path curvature and initial conditions that guarantee a bounded change in state between impacts. However, it is not clear how to derive the bounds for this form of stability (e.g., the maximum curvature safely allowed from some initial state).

This paper contributes a pair of *a priori* rules that ensure stable sequential composition from a set of asymptotically stable gaits derived from *any* low-level controller. We use switched systems theory to prove these rules, considering the more general case of *locally asymptotically* stable systems as opposed to *globally exponentially* stable systems in [14]. This stability result reduces a high-dimensional control and planning problem to a low-dimensional discrete tree search, where paths through the workspace correspond to composite (Lyapunov) funnels that obey the rules admitted by a *small* set of controllers. This differs from the pioneering funneling work [15], in which a workspace is robustified by covering it with regions of attraction from *many* locally stabilizing controllers. We extend our preliminary work [16] by (1) implementing a planning algorithm around our two rules, and (2) considering more complex biped models. Hence, we combine control and planning into one coherent approach–we are unaware of other planning results for asymptotically stable 3D walking.

We begin by describing the 3D compass-gait biped in Section II, which we use as a canonical example to construct our planning framework. We formalize the notion of *asymptotically stable gait primitives* and the planning problem

Manuscript received 3/2/12. *Asterisk indicates corresponding author.*
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This research was supported by NSF Grants CMMI-0856368, CMS-0510119, and NSF-0931871.

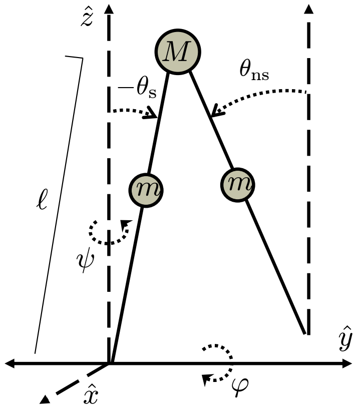

Fig. 1. Three-dimensional extension of the compass-gait biped.

admitted by such primitives in Section III. We prove bounds on steering curvature and switching frequency that allow stable composition of gait primitives in Section IV, implying that a walking path composed of these primitives may be stably followed by the robot. We derive a set of primitives and composition rules for the compass-gait biped (using controlled reduction) in Section V. We implement a discrete search algorithm in Section VI to plan open-loop primitive sequences for walking through workspaces with obstacles, which we then extend to an underactuated five-link 3D biped. We conclude with remarks and future work in Section VII.

II. Bipedal Walking as a Hybrid System

In order to study locomotion with impulsive impacts, we must consider both continuous and discrete dynamics in a hybrid system. Bipedal walking gaits correspond to hybrid limit cycles that are stable from step to step. We now use the canonical example of the compass-gait biped to construct the formalisms necessary for our planning framework.

The 3D extension of the planar compass-gait biped is shown in Fig. 1. The generalized configuration space of this two-link model is $\mathrm{SE}(3)\times\mathbb{S}^1$, where the stance foot has six DOFs (three position, three orientation) and the hip has one rotational DOF. Assuming the ground has sufficient friction and the stance foot is sufficiently large to remain in contact without slipping or rotating (i.e., fixed cartesian coordinates), we can consider the reduced configuration space $\mathcal{Q} = \mathrm{SO}(3) \times \mathbb{S}^1$ for the continuous dynamics of single-support phase. We parameterize this space with the coordinate vector $q = (\psi, \varphi, \theta^T)^T \in \mathbb{T}^4$, where $\psi, \varphi \in \mathbb{S}^1$ are respectively the heading/yaw and roll/lean variables at the stance foot, and sagittal-plane vector $\theta = (\theta_{\rm s}, \theta_{\rm ns})^T \in \mathbb{T}^2$ contains the stance and swing leg pitch variables. This choice of coordinates will allow us to consider dynamical stability independent of the cartesian workspace.

This simple biped has no hip link, so each leg has identical single-support dynamics (i.e., gaits will consist of one step cycle). The system state is given by $x = (q^T, \dot{q}^T)^T$ in phase space $T\mathcal{Q}$, where vector $\dot{q} \in \mathbb{R}^4$ contains the joint velocities. Foot-ground impacts are the only discrete events, which are instantaneous and perfectly plastic, so we define a simple hybrid dynamical system with one continuous phase:

$$\begin{array}{ll} \dot{x} = f(x) + g(x)u & x \in D\backslash G \\ x^+ = \Delta(x^-) & x^- \in G \end{array}, \qquad (1)$$

where $u \in \mathbb{R}^m$ is the control input vector for $m \leq 4$, domain $D \subset T\mathcal{Q}$ is the set of states with nonnegative swing foot height, switching set $G \subset D$ contains states where the swing foot height is zero and decreasing (a hyperplane in $T\mathcal{Q}$), and $\Delta : G \to D$ is the reset map modeling the discontinuous impact event. Details on these dynamics can be found in [12].

Given a state-feedback controller for input u, (1) becomes a closed-loop, time-invariant hybrid system that is solved by a curve $x(\cdot) : \mathbb{R}_+ \to T\mathcal{Q}$ called a *hybrid flow*. An appropriately designed controller will produce walking gaits that correspond to *periodic* hybrid flows such that $x(t) = x(t+T)$ for all $t \geq 0$ and some minimal $T > 0$ known as the time-to-impact between discrete events. When the image of a periodic flow in $T\mathcal{Q}$ is an isolated orbit, it is known as a *hybrid limit cycle*.

Hybrid limit cycles correspond to equilibria of the return map $P : G \to G$, which represents a hybrid system as a discrete system between impact events. This map sends state $x_i \in G$ ahead one step to $x_{i+1} = P(x_i)$, so a period-one hybrid limit cycle has a fixed point $x^* = P(x^*)$. We say that x^* is *stable* if for each $\epsilon > 0$, there exists a constant $\gamma > 0$ such that for all sequences $\{x_i\}$ with $|x_0 - x^*| \leq \gamma$, $|x_i - x^*| \leq \epsilon$ for all $i \geq 0$. A fixed point is *locally asymptotically stable* (LAS) if it is stable *and* $|x_i - x^*| \to 0$ as $i \to \infty$.

We know that a hybrid limit cycle is LAS if its associated fixed point is LAS [1]. We verify LAS of x^* by computing the linearized map $\nabla_x P(x^*)$ in simulation [8]. This yields a discrete linear system that is asymptotically stable if the magnitudes of the eigenvalues of $\nabla_x P(x^*)$ are strictly less than one. The local stability region about fixed point x^*, known as the *basin of attraction*, is then

$$BoA(x^*) \quad = \quad \left\{ x \in G \ \text{ s.t. } \lim_{i\to\infty} P^i(x) = x^* \right\}. \qquad (2)$$

A walking gait corresponds to a LAS fixed point of return map P in local coordinates $x \in T\mathcal{Q}$. Turning gaits have a fixed change in heading s, resulting in a fixed point *modulo yaw*

$$x^*(s) + (s \quad 0_{1\times 7})^T \quad = \quad P_s(x^*(s)). \qquad (3)$$

Straight-ahead gaits are then asymptotically stable about a zero steering angle. We will express fixed points as a function of steering angle s throughout the rest of the paper.

III. Asymptotically Stable Gait Primitives

We now use this construction to define asymptotically stable motion primitives. We will later see that this concept generalizes to more complex bipeds that admit LAS hybrid limit cycles, since hybrid mechanical systems can be defined as discrete systems using the method of Poincaré sections [1].

In the context of path planning we must consider the robot's world coordinates in the generalized configuration space $\mathrm{SE}(3) \times \mathbb{S}^1$. For walking on a flat surface, we need only measure the $\mathrm{SE}(2)$ coordinates of stance foot position $p \in \mathbb{R}^2$ and heading $\psi \in \mathbb{S}^1$ (the first term in configuration vector q) at every step. The biped's extended state is then $x^{\rm e} = (p^T, x^T)^T \in \mathbb{R}^2 \times T\mathcal{Q}$ with extended return map $P^{\rm e}$, which updates positions using forward kinematics.

Definition 1: Given a coordinate parameterization $h \in \mathrm{SE}(2)$, define the group action

$$\Phi_h(x^{\mathrm{e}}) = (h^T + (p^T, \psi), \varphi, \theta^T, \dot{q}^T)^T. \quad (4)$$

Extended map P^{e} is *equivariant* under SE(2) if for all $h \in \mathrm{SE}(2)$ and $x^{\mathrm{e}} \in \mathbb{R}^2 \times T\mathcal{Q}$, $\Phi_h \circ P^{\mathrm{e}}(x^{\mathrm{e}}) = P^{\mathrm{e}} \circ \Phi_h(x^{\mathrm{e}})$.

This symmetry property in the global coordinates is guaranteed on level ground when using a control law that is independent of heading ψ (based on SO(3)-invariance [17]). We can now formalize the notion of asymptotically stable motion primitives for 3D walking through flat environments.

Definition 2: An *asymptotically stable gait primitive* with steering angle s is a pair $\mathcal{G}(s) = (P_s, x^*(s))$ such that (3) is satisfied and the extended map P_s^{e} is equivariant under SE(2).

The extended map of a gait primitive yields a fixed point in local coordinates and a walking arc in SE(2). Transient effects from gait switching (i.e., converging back and forth between differing orbits) prevent a fixed mapping from gaits to path arcs during a walking sequence, but each gait segment is approximated by a constant-curvature arc.

Definition 3: The *nominal walking arc* of primitive $\mathcal{G}(s)$ is the pair $\left(\delta p^T(s), s\right)^T \in \mathrm{SE}(2)$ with position displacement $\delta p(s) \in \mathbb{R}^2$ and heading change s from initial heading $\psi = 0$.

Walking arcs are composed with different orientations by rotating the nominal arc's coordinate frame to coincide with initial heading ψ, i.e., $R_\psi \delta p(s)$ where $R_\psi \in \mathrm{SO}(2)$ is the standard rotation matrix with respect to angle ψ.

The *basis* set $\mathcal{P}(s) = \{\mathcal{G}(0), \mathcal{G}(s), \mathcal{G}(-s)\}$ parameterized by steering angle s contains three primitives: straight-ahead, clockwise (CW), and counter-clockwise (CCW). This set grows a discrete tree of nominal arcs with branching factor three. If we derive *a priori* rules that ensure stable composition between primitives in a set $\mathcal{P}(s)$, we will have reduced a complicated kinodynamic planning problem in $\mathbb{R}^2 \times T\mathcal{Q}$ to a discrete search in SE(2). The planning algorithm can then be designed to output a sequence of steering angles parameterizing gait primitives (e.g., by preimage backchaining from the goal position [18]). The process flow diagram for this hierarchical planning framework is shown in Fig. 2.

In this construct the output of the planner drives the event-based control $\sigma(\cdot)$ in the discrete switched system

$$x_{i+1}^{\mathrm{e}} = P_{\sigma(i)}^{\mathrm{e}}\left(x_i^{\mathrm{e}}\right), \quad (5)$$

where, at each impact event i, switching signal $\sigma : \mathbb{Z}_+ \to \{0, s, -s\}$ chooses a closed-loop system $P_{\sigma(i)}^{\mathrm{e}}$ parameterized by the steering angle of a primitive. We now derive constraints on angle s and signal σ that ensure stability of (5).

IV. Rules for Sequential Composition

We now present a switched system formulation of the funneling approach to controller composition [15], by which we derive our main technical contribution: an upper bound on the steering angle s parameterizing basis set $\mathcal{P}(s)$, and a lower bound on the switching frequency of σ in (5). These *a priori* rules will ensure stable composition of gait primitives for an n-DOF biped ($n = 4$ for our example), where we do not necessarily have a symbolic expression for the return map.

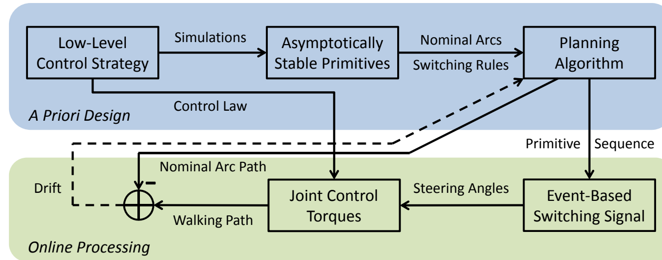

Fig. 2. Process flow diagram of planning framework. Dashed line represents feedback that would be needed for iterative planning to compensate for drift.

Definition 4: Step cycle i is represented by the pair $(x_{i-1}, \mathcal{G}_i)$, where $\mathcal{G}_i$ is the gait primitive employed after the $i-1^{th}$ impact event with impact state x_{i-1}. Step cycle i is said to be *switching* if $\mathcal{G}_i \neq \mathcal{G}_{i-1}$ and *stable* if $x_{i-1} \in BoA(x_i^*)$, where x_i^* is the LAS fixed point of primitive $\mathcal{G}_i$.

Step cycle $i+1$ is then related to cycle i by $x_i = P_{\sigma(i)}(x_{i-1})$. Invariance of the basin of attraction implies that step cycle $i+1$ is stable if cycle i is stable and $\mathcal{G}_{i+1} = \mathcal{G}_i$. We derive the first rule for stable composition by exploiting a convergence property of continuously parameterized gaits:

Assumption 1: For every steering angle $s \in [-S, S]$ for some $S > 0$, there exists LAS fixed point $x^*(s)$ of P_s with corresponding $BoA(x^*(s))$. Then, by definition there exists a non-empty open ball of radius $r(s) > 0$ about $x^*(s)$ such that $\mathcal{B}(x^*(s), r(s)) \subset BoA(x^*(s))$, where $x^*(s)$ and $r(s)$ are assumed continuous functions of s.

Property 1: Continuity of function $x^*(s)$ implies convergence to fixed point $x^*(0)$ in metric space $(\mathbb{R}^{2n}, d)$ as $s \to 0$, i.e., $\lim_{s\to 0} d(x^*(s), x^*(0)) = 0$ for Euclidean distance d.

Turning motion more closely resembles straight-ahead motion for smaller steering angles. We then can exploit overlap in neighboring basins of attraction to derive our first rule.

Rule 1: The absolute turning curvature $|s|$ of basis $\mathcal{P}(s)$ is bounded above by some $\tilde{S}$ satisfying Lemma 1.

Lemma 1: There exists a positive steering angle $\tilde{S} \leq S$ such that for all $s \in [-\tilde{S}, \tilde{S}]$:

1) $x^*(0) \in \mathcal{B}(x^*(s), r(s)) \subset BoA(x^*(s))$
2) $x^*(s) \in \mathcal{B}(x^*(0), r(0)) \subset BoA(x^*(0))$
3) $x^*(-s) \in \mathcal{B}(x^*(s), r(s)) \subset BoA(x^*(s))$

Proof: [1.1] We first define minimal ball radius $\underline{r} := \min_{\hat{s}\in[-S,S]}(r(\hat{s}))$, positive by compactness of $[-S, S]$, so

$$\mathcal{B}(x^*(s), \underline{r}) \subset \mathcal{B}(x^*(s), r(s)) \subset BoA(x^*(s))$$

for all $s \in [-S, S]$. Since $\underline{r} > 0$ and $\lim_{s\to 0} d(x^*(s), x^*(0)) = 0$, $\exists\ \tilde{S} \leq S$ such that $d(x^*(s), x^*(0)) < \underline{r}$ for all $s \in [-\tilde{S}, \tilde{S}]$. Then $x^*(0) \in \mathcal{B}(x^*(s), \underline{r})$ for all $s \in [-\tilde{S}, \tilde{S}]$.

[1.2] Similarly, $x^*(s) \in \mathcal{B}(x^*(0), \underline{r})$ for all $s \in [-\tilde{S}, \tilde{S}]$.

[1.3] Recall $x^*(s) \to x^*(0)$ as $s \to 0$, which means that for each $\epsilon/2 > 0$, $\exists\ \delta > 0$ such that for all $s \in [-\delta, \delta]$, $d(x^*(s), x^*(0)) < \epsilon/2$. Then, the triangle inequality shows

$$d(x^*(s), x^*(-s)) \leq d(x^*(s), x^*(0)) + d(x^*(-s), x^*(0)) < \epsilon.$$

Hence, $\lim_{s\to 0} d(x^*(s), x^*(-s)) = 0$.

As in 1.1, $\exists\ \tilde{S}$ such that $d(x^*(s), x^*(-s)) < \underline{r}$ for all $s \in [-\tilde{S}, \tilde{S}]$. Then, $x^*(-s) \in \mathcal{B}(x^*(s), \underline{r})$ for all $s \in [-\tilde{S}, \tilde{S}]$.

Finally, we take the smallest $\tilde{S}$ from the three proofs. ■

Remark 1: If contained in open ball $\mathcal{B}(x^*(s), r(s))$, $x^*(0)$ cannot be on the boundary of $BoA(x^*(s))$. Therefore, points sufficiently close to $x^*(0)$ are also contained in $BoA(x^*(s))$. The same holds for the other claims in Lemma 1.

Although a converging trajectory never reaches a fixed point in finite time, the state will eventually be close enough for switching. If switching signal σ in (5) provides a sufficiently long *dwell time* for a primitive, the state will be funneled into the basins of attraction of the other primitives. Our second rule therefore constrains this signal to ensure stable composition.

Rule 2: The switching signal σ of (5) has a minimum dwell time $N \geq 1$, i.e., $\sigma(i+j) = \sigma(i)$ for all $j \leq N$ and all i such that $\sigma(i-1) \neq \sigma(i)$, where N satisfies Theorem 1.

Theorem 1: For any $s \in [-\tilde{S}, \tilde{S}]$ there exists a minimal number of steps $N \geq 1$ such that for all integers $k \geq N$:

1) If $x \in \mathcal{B}(x^*(0), r(0))$, then $P_0^k(x) \in \mathcal{B}(x^*(s), r(s))$
2) If $x \in \mathcal{B}(x^*(s), r(s))$, then $P_s^k(x) \in \mathcal{B}(x^*(0), r(0))$
3) If $x \in \mathcal{B}(x^*(s), r(s))$, then $P_s^k(x) \in \mathcal{B}(x^*(-s), r(-s))$

Proof: We know from Lemma 1 and Remark 1 that for any pair $\hat{s}, \bar{s} \in [-\tilde{S}, \tilde{S}]^2$, $\exists\ \bar{r}(\hat{s}, \bar{s}) > 0$ (assumed continuous) such that $\mathcal{B}(x^*(\hat{s}), \bar{r}(\hat{s}, \bar{s})) \subset \mathcal{B}(x^*(\bar{s}), r(\bar{s}))$. By definition of LAS, for every $\epsilon > 0$ and $x \in BoA(x^*(\hat{s}))$, $\exists\ N_{\epsilon,x}(\hat{s}) \geq 1$ such that for every $k \geq N_{\epsilon,x}(\hat{s})$, $d(P_{\hat{s}}^k(x), x^*(\hat{s})) < \epsilon$. Letting $\epsilon = \min_{\hat{s},\bar{s} \in [-\tilde{S},\tilde{S}]^2} \bar{r}(\hat{s}, \bar{s})$, positive by compactness of $[-\tilde{S}, \tilde{S}]^2$, $N = \max_{\hat{s} \in \{0,s,-s\}} \{\sup_{x \in \mathcal{B}(x^*(\hat{s}), r(\hat{s}))} N_{\epsilon,x}(\hat{s})\}$ is finite by completeness of $\mathbb{R}$ and satisfies claims 1-3. ■

Note that the minimum dwell time N is conservative, so violating this rule does not necessarily imply instability. We may not always be able to explicitly compute the basins of attraction, so in Section V we propose a simulation-intensive heuristic to estimate the value of N. We will also numerically verify that the absolute steering angle is bounded above by $\tilde{S}$.

We have shown that the stability of a walking sequence is ensured *a priori* by two rules: an *upper bound on absolute curvature* and a *lower bound on dwell time*. A hierarchical controller (e.g., a finite-state machine) for signal σ can then piece together straight and curved segments such that the turns are not too sharp or the primitive switches too frequent. By considering *locally asymptotically* stable systems, this result is more broadly applicable than the dwell time derived for *globally exponentially* stable systems (a stronger form of LAS) in [14]. Our switching framework is also related to the aperiodic sense of stability considered in [2]. We now construct an example set of LAS gait primitives.

V. Compass-Gait Biped Example

We now apply this framework to the canonical example of the compass-gait biped. We will later demonstrate that the framework generalizes to more complex models that admit LAS gaits (e.g., [6], [13], [19], [20]).

The first step is to choose a low-level controller that stabilizes a set of gaits for the hierarchical system in Fig. 3.

A. Choosing a Stabilizing Low-Level Controller

For simplicity we assume full actuation ($m = n = 4$) in our compass-gait model, but we will consider an underactuated

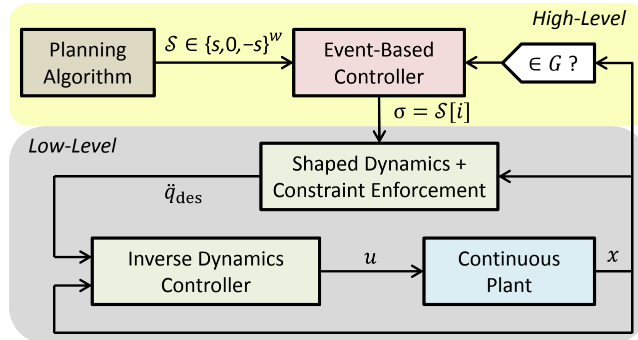

Fig. 3. Low-level and high-level control loops. Note that G is the switching set for ground strike, w is the number of primitives in a planned sequence, σ is the switching signal, and $\ddot{q}_{\text{des}}$ are the closed-loop joint accelerations.

model in Section VI. The control input u is subject to actuator saturation, where $|u_j| \leq U^{\max}$ for all $j \in \{1, \ldots, 4\}$.

Reduction-based control exploits the existence of momentum conservation laws to decompose robot dynamics into lower-dimensional control problems [11], [13]. These laws are nonholonomic constraints that can be controlled as in [13], [20] to drive yaw toward desired heading $\bar{\psi}$ and stabilize lean about vertical $\bar{\varphi} = 0$. A geometric reduction with respect to these conservation laws defines a projection onto a reduced-order system corresponding to the decoupled sagittal plane (the $\hat{y}$-$\hat{z}$ plane of Fig. 1). Gaits are then constructed from passively stable periodic motions in the sagittal plane (e.g., [17]).

A feedback control law is designed for this purpose in [11], which inverts the plant and reinserts the original dynamics plus shaping and constraint enforcement terms (see [11], [13] for details). This controller will interact with the planner as shown in Fig. 3, but we first must construct a set of gait primitives.

B. Constructing the Primitive Set

We assign common physical parameters from the literature to construct our primitive set: M = 10 kg, m = 5 kg, ℓ = 1 m, $U^{\max}$ = 20 Nm. The closed-loop hybrid system yields straight-ahead walking on flat ground by setting $\bar{\psi} = 0$ (without loss of generality) to find the fixed point $x^*(0)$ as in [12]. We numerically verify LAS of this straight-ahead gait by linearizing the associated Poincaré map P_0. This defines the straight-ahead gait primitive $\mathcal{G}(0) = (P_0, x^*(0))$, which has a periodic step length of 0.534 m and speed of 0.727 m/s.

We create turning gaits by steering with a constant angle s between steps, where the event-based controller in (5) increments desired yaw $\bar{\psi}$ by s at each impact event [12]. We want to show that for sufficiently small s, trajectories of the hybrid system converge to a period-one orbit corresponding to an LAS fixed point (modulo yaw) of P_s. We can then define CW-turning and CCW-turning gait primitives $\mathcal{G}(s)$ and $\mathcal{G}(-s)$, which have mirroring orbits in the yaw/lean coordinates.

Initialized at $x^*(0)$, we observe convergence to a fixed point $x^*(s)$ for any choice of $s \in [-S, S]$, $S = 0.492$. We densely sample steering values in $[-S, S]$, find the fixed point for each sample, and confirm LAS as numerical evidence of Assump. 1 and Property 1. Input-to-state stability provides that trajectories will remain nearby for steering values between sufficiently

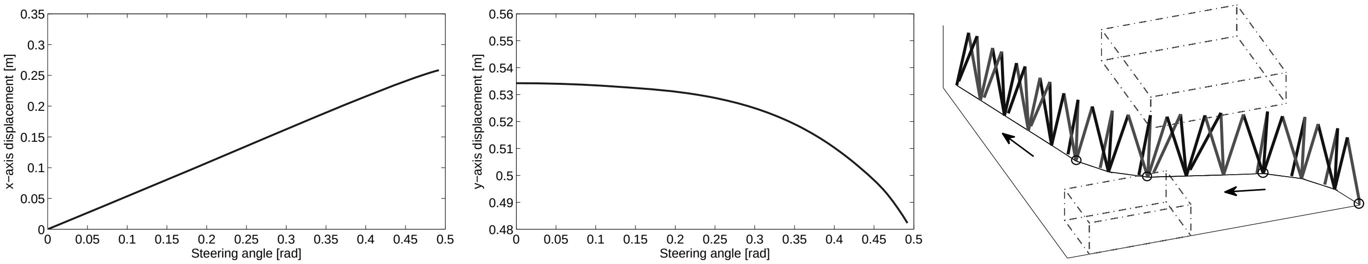

Fig. 4. Evolution of $\hat{x}$-axis (left) and $\hat{y}$-axis (right) displacements over steering angle $s \in [0, 0.492]$. Animation of an example planned walking sequence (right). The sequence of primitives is (CCW, CCW, CCW, S, S, S, CW, CW, S, S, S, S, S), where switching steps are indicated by circles at impact events.

TABLE I

Primitive	Command	$\delta p_1(s)$ [m]	$\delta p_2(s)$ [m]	s [rad]
Straight	$\bar{\psi} = \psi$	0	0.534	0
CW	$\bar{\psi} = \psi + 0.32$	0.1733	0.5229	0.32
CCW	$\bar{\psi} = \psi - 0.32$	-0.1733	0.5229	-0.32

dense samples (arguably with LAS fixed points). The position displacements for the nominal walking arc associated with each steering angle are given in Fig. 4.

Although it is difficult to find the exact steering bound $\tilde{S}$ for Rule 1, we can easily verify the conditions of Lemma 1 for some s by checking convergence from all fixed points of basis set $\mathcal{P}(s)$. We confirm that Rule 1 is satisfied for $\tilde{s} = 0.32$ (i.e., $\tilde{s} \leq \tilde{S}$). The nominal walking arcs of basis set $\mathcal{P}(\tilde{s})$ are given in Table I (refer to [12] for the fixed points). We also verify that these simulated gaits do not violate unilateral ground contact constraints by calculating the ground reaction forces as in [13].

We now must numerically derive the minimum dwell time of Rule 2 for this basis set of primitives.

C. Computing the Switching Bound

The overlapping region between basins of attraction influences the minimum dwell time N for Theorem 1. If the biped's transient state leaves this safe region, an unstable switching scenario becomes possible. In particular, high-frequency switching may not provide sufficient time for a gait to attenuate transients. Eventually, the impact-event state from one gait primitive may be outside the basin of attraction of the next.

We attempt to deduce N for $\mathcal{P}(\tilde{s})$ by exhaustively testing gait switching scenarios with a "random walk," picking a gait primitive every step from a uniform random distribution. We observe occasional falls after dozens of steps, implying that $N > 1$. We next allow switching every other step and are unable to produce falls after several lengthy simulations (1000+ steps), suggesting that minimum dwell time $N = 2$.

This simulation-intensive procedure explores a lower-dimensional discrete space of sequences rather than the full continuous state space, which allows application to high-dimensional bipeds. The resulting estimate is not rigorous, but we have shown that falling scenarios are rare for the states commonly encountered during walking. Emerging work on transverse dynamics and sum-of-squares verification for estimating basins of attraction of hybrid limit cycles [21] may prove essential for rigorous bounds on dwell time [14].

These simulations provide evidence that the overlapping attractive region of the primitive set is large, due to the close proximity of the fixed points as well as the large sizes of the associated basins of attraction. Hence, this primitive set is capable of building a large class of stable walking paths, enabling planning through workspaces (animated in Fig. 4).

VI. Path Planning Applications

The gait primitive framework (depicted in Fig. 2) provides a layer of abstraction above the low-level control and stability of a walking mechanism to enable motion planning by switching between pre-stabilized gaits. We can compose these separate primitives in discrete pieces to generate trajectories that simultaneously perform obstacle avoidance and direct the robot to a goal region in the workspace. In this section, we present one possible approach to planning based on gait primitives as a proof of concept. Note however that the development of this paper opens the possibility of a wide variety of planning algorithms being used in real robot systems.

We begin with a basis primitive set, which corresponds to a discrete subset of the continuous range of steering angles available to the biped. The set of possible paths (concatenations of motion primitives) can then be characterized by a tree data structure with branching factor equal to the cardinality of the primitive set. The number of paths encoded in this tree expands exponentially as the number d of concatenated primitives in a path grows. Thus, the tree will represent $O(3^d)$ paths composed of nominal walking arcs. If d is large (the path length to be walked is long) and we wish to choose a path based on certain criteria, i.e., collision-free, reaches the goal, and minimizes a cost function, we will need to heuristically bias the exploration of the search tree. This is a well-studied problem [22]–[24], and in particular the A* graph search algorithm is frequently applied to humanoid robots, e.g., [10], [25]. We will use a variant of [25] to plan our walking paths.

We define initial world configuration $c_0 \in \mathrm{SE}(2)$ and goal region $F \subset \mathrm{SE}(2)$ so that any stable walking sequence ending at a world configuration $c_w \in F$ after some number of steps w is considered admissible. The planner outputs a steering sequence $\mathcal{S}_w \in \{0, s, -s\}^w$ corresponding to the gait primitive for each step in the walking path. This sequence is designed to produce a trajectory in system (5) that is collision free and terminates in the goal region while minimizing a scalarized objective function $\mathcal{C}[\mathcal{S}_w]$ that penalizes nominal path length and number of gait switches (according to weight factor α):

$$\mathcal{C}[\mathcal{S}_w] = \sum_{i=1}^{w} \mathrm{norm}(\delta p(\mathcal{S}_w(i))) + \alpha \mathbb{1}\{\mathcal{S}_w(i) \neq \mathcal{S}_w(i-1)\}.$$

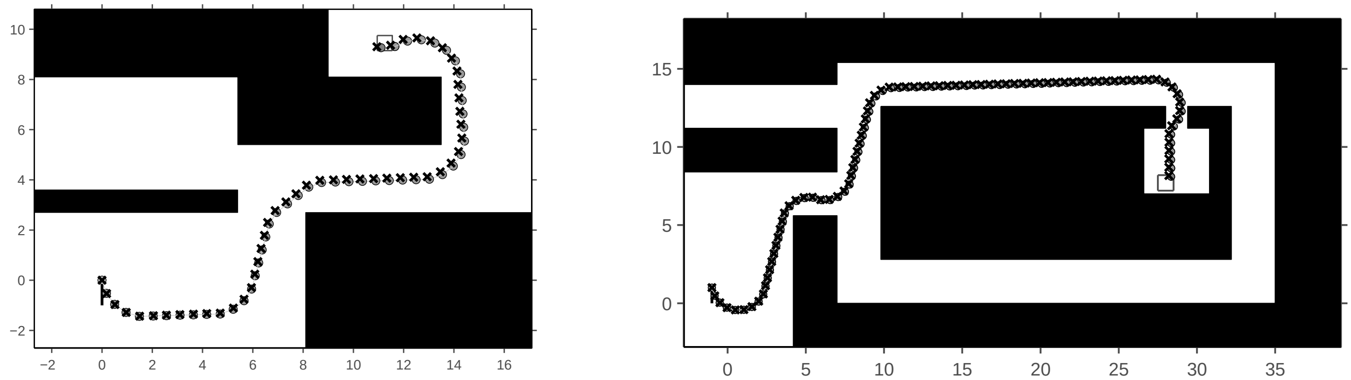

Fig. 5. Two planned walking environments with basis set $\mathcal{P}(\tilde{s})$, $\tilde{s} = 0.32$. Planned steps are indicated by gray circles and simulated steps by black X's. Initial orientation is shown by a black line from the starting position. Maximum, mean, and final drift values are respectively 0.197, 0.107, and 0.169 m (left) and 0.168, 0.105, and 0.168 m (right). Supplementary downloadable videos are available for these planned walking cases.

The planner begins by performing a workspace decomposition that bounds obstacles with safety regions and decomposes the free space into a set of convex cells. We use the shortest path on the workspace skeleton[1] to identify the path homotopy class[2] we will explore to find our path composed of motion primitives. This heuristic works well in practice for reducing the number of paths to be explored without removing desirable paths [25]. Our second complexity-reducing approximation is a branch and bound style of tree search, where we plan optimal sequences of motion primitives between subgoal regions in the configuration space. We identify these subgoals by finding intersections between workspace curves corresponding to the projection of paths belonging to our selected homotopy class and boundaries of the workspace decomposition.

We employ the A^* Algorithm to compute optimal paths between subgoals. This algorithm expands the search tree by choosing nodes from a priority queue that is sorted by the *cost-to-come*, a partial computation of $\mathcal{C}$ (on the segment of the primitive sequence explored thus far), plus the estimated *cost-to-go*. The true cost-to-go from a node, i.e., the minimal cost to reach the goal set from the configuration of that node, is not known until the algorithm completes, so it is approximated with a heuristic function that lower bounds the true cost-to-go. For this we use the Euclidean distance between workspace projections of the robot's configuration at the node and the closest point in the next subgoal region. Sequences of motion primitives that violate dwell time constraints or likely cause obstacle collisions are pruned during the process of node expansion. When A^* terminates, we have a path plan from one subgoal region to the next. We can then start a new search from the terminal configuration of this path to the next subgoal region. The final subgoal region is identical to the goal region, by which the plan generation is completed.

A. Compass-Gait Biped Results

We use the primitive basis set $\mathcal{P}(\tilde{s})$ of Table I with $\tilde{s} = 0.32$. Setting weight factor $\alpha = 0.6$, the planner takes seconds to produce the nominal paths shown in gray in the example environments of Fig. 5. The compass-gait biped is then simulated with the corresponding sequence of primitives, resulting in a walking path (shown in black) that traces the pre-planned path into the goal region with only minor drift. In both cases, the average and final errors from planned step placements are respectively 20% and 32% of one step length.

Recall that the biped does not explicitly track this planned path. The nominal walking arcs associated with the open-loop primitive sequence accurately predict the simulated walking path. This is noteworthy given the transient effects after each switching step. Hundreds of randomly generated paths show that final drift is more strongly correlated with the number of switches than steps (determination coefficients of 0.64 vs. 0.28). Occasional re-planning as suggested in Fig. 2 can easily compensate for drift accumulation after many switches/steps.

B. Five-Link Biped Results

This planning framework can be applied to a large class of bipedal walkers that admit hybrid limit cycles satisfying Property 1. Model complexity and underactuation only challenges the low-level control design. We now demonstrate how the planner generalizes to more complex models by considering the underactuated five-link 3D biped of [19] shown in Fig. 6.

Given the same foot contact assumptions for the compass-gait biped, the hybrid system of the five-link biped has two distinct phases during single support: knee-swing with six DOFs and knee-lock with five DOFs. We assume that both knee-strike and ground-strike impact events (respectively triggered by sets G_{k} and G_{g}) are instantaneous and perfectly plastic, resulting in transitions between the six and five DOF dynamics according to Fig. 6. The knee of the stance leg remains locked through the entire single-support cycle. The yaw DOF at the ankle is unactuated but subject to viscous damping from friction (which stabilizes this motion [13]). The other DOFs including ankle lean and pitch are actuated ($m = n - 1$) with a torque bound of 40 Nm. We adopt the underactuated formulation of reduction-based control from [13] to construct a set of two-step gait primitives for this 35 kg biped, where heading is controlled by leaning into turns.

We characterize the state evolution over a two-step gait cycle by twice composing the return map $P : G_{\mathrm{g}} \to G_{\mathrm{g}}$. The

[1]The workspace skeleton is a deformation retract of the free workspace, implemented as a discrete approximation of the generalized Voronoi diagram.

[2]Two paths that are homotopic to one another are identical after a homotopic transformation corresponding to a deformation retract [23].

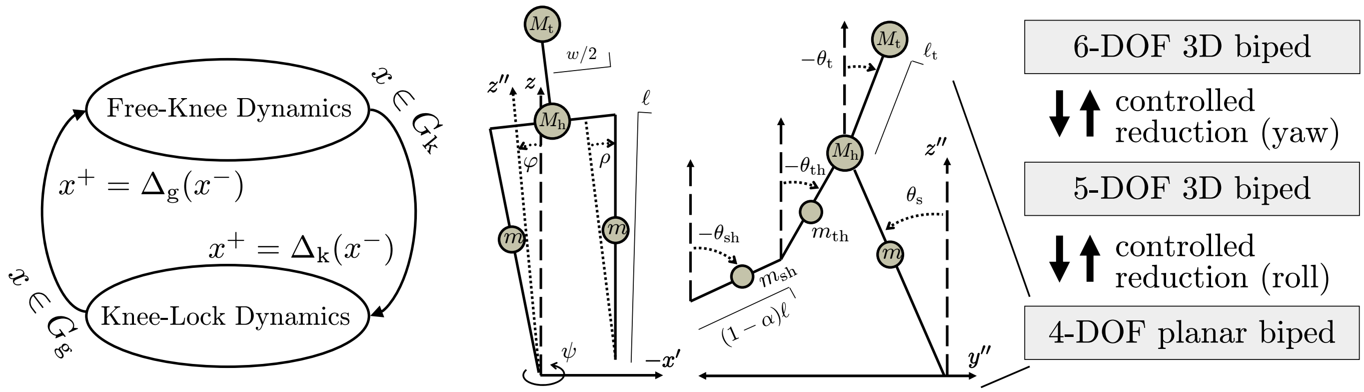

Fig. 6. Diagrams of the five-link 3D biped's hybrid system (left), frontal and sagittal planes (middle), and controlled reduction (right). The control strategy decouples the dynamics of the sagittal plane by reducing the yaw DOF of the transverse plane and the roll/lean DOF of the frontal plane.

TABLE II

Primitive	Command	δp_1 [m]	δp_2 [m]	s [rad]
Straight	$\bar{\varphi} = 0$	0	1.1970	0
CW	$\bar{\varphi} = -0.009$	0.1962	1.1751	0.1968
CCW	$\bar{\varphi} = 0.009$	-0.1962	1.1751	-0.1968

Poincaré map P^2 gives us fixed points as before, reducing this complex hybrid system to the simple construction of Sections III-IV. Due to the unactuated yaw DOF, the steering angle $s = s(\bar{\varphi})$ is now a function of the lean set-point. Fig. 7 shows that this (odd) function is continuous and one-to-one, allowing satisfaction of Property 1 as before.

The basis set of primitives $\mathcal{P}(s(\bar{\varphi}))$ derived from $\bar{\varphi} = 0.009$ is given in Table II. These primitives satisfy Lemma 1.1-1.2 and Theorem 1.1-1.2, so switches are possible between turning and straight-ahead gaits but not necessarily between opposing (CW/CCW) turning gaits. We estimate the minimum dwell time for these relaxed conditions following the procedure of Section V-C and find that switching between straight-ahead and turning is allowed every gait cycle ($N = 1$). Integrating instantaneous power to compute the work done per unit weight per unit distance, we find that the mechanical cost of transport for each primitive is small at 0.037 and similar to passive dynamic walkers such as the Cornell biped at 0.055 [26].

The only modification needed in the previously described planning algorithm is that branches corresponding to switches between CW/CCW are also pruned during node expansion. We expect greater transient drift from pre-planned walking paths due to this model's lack of direct steering control, so we expand the safety boundaries around obstacles and further penalize switches with $\alpha = 1$. Before simulating the path output by the planner, we delete the first two primitives (and twice repeat the last) in the sequence so that the biped employs each gait primitive two cycles early. This causes leaning ahead of planned turns to further reduce drift from the desired path.

Fig. 8 shows the planned and simulated paths of the (enlarged) first example environment. The drift from the desired destination is 2.59 m after an open-loop sequence of 104 steps, which can be improved with an iterative planning algorithm. The fact that our high-level framework is applicable to complex biped models without direct control over steering is a testament to the significance of the approach. We have enabled

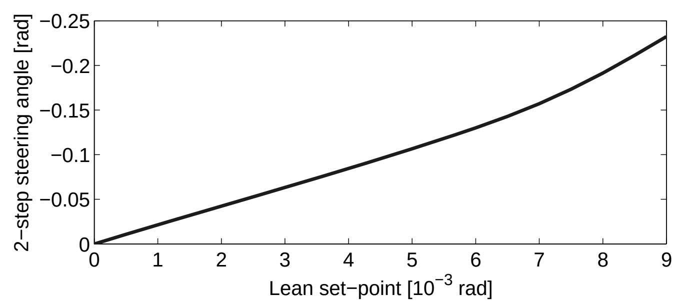

Fig. 7. Five-link biped: two-step steering angle of steady-state turning gait against lean set-point $\bar{\varphi}$ of the inner leg's stance phase. This curve is an odd function of the lean set-point, i.e., negating the $\hat{x}$-axis negates the $\hat{y}$-axis.

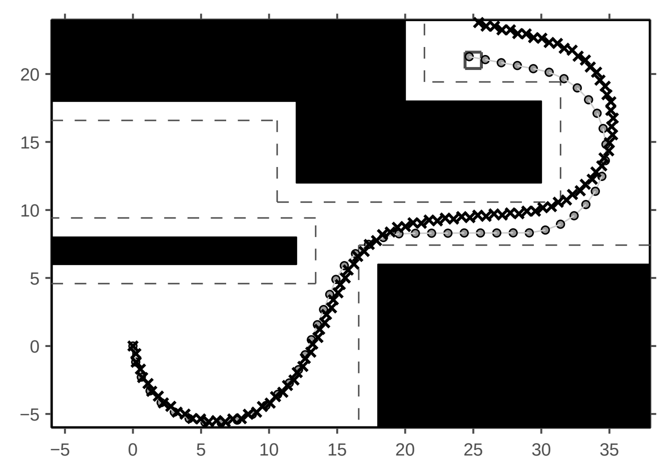

Fig. 8. Five-link biped: planned walking with primitive set $\mathcal{P}(\bar{\varphi})$, $\bar{\varphi} = 0.009$. Nominal primitive arcs are connected by gray circles (representing the second step of a gait cycle) and simulated steps are indicated by black X's. Obstacle safety region of planner delineated by dashed boundaries. The drift from the desired destination is 2.59 m after an open-loop sequence of 104 steps.

dynamic walking through workspaces for 3D robots that have similar energetic efficiency to passive dynamic walkers.

VII. Conclusions

We reduced a complicated feedback motion planning problem in a high-dimensional state space to a much simpler discrete path planning problem with a low-dimensional characterization of the robot's configuration. This allows decomposed planning of asymptotically stable bipedal walking using search tools developed for ZMP walking [9], [10], [27], [28].

Our planning framework can potentially be used with any form of locomotion based on asymptotically stable gaits (e.g., walking [1], [6], [13], [20], brachiating [4], climbing [5], or running [1], [29]). Each gait primitive is characterized by a stabilizing controller and a fixed point, where the associated hybrid limit cycle is LAS in the closed-loop robot dynamics. Walking motion is not prescribed by full-state trajectories or subjected to postural ZMP constraints, yet we have stability over a large class of paths composed of gait primitives.

In order to reach specific goal configurations, future work might generalize this framework to allow primitive switching within the full continuous range of available steering angles. Gait primitives and their stability rules might also be pre-computed using the feedback motion planning method of randomized LQR trees [30] with sum-of-squares programming [21]. Practical implementations of the gait primitive framework could integrate a suite of other feedback motion planning tools, such as step-level planning over rough terrain [31]–[33] and time-scaling for variable walking speeds [34].

Asymptotically stable walking has been experimentally demonstrated on planar bipeds (e.g, [1]–[3]), and 3D results will soon be possible with advances in controller and hardware design. The yaw DOF in our five-link biped can be controlled passively with viscous damping [13], allowing more feasible 2-DOF ankle actuation for lean and pitch. Preliminary work has implemented controlled reduction on a 16-DOF Sarcos humanoid model to achieve stable balancing [20] and on the NAO robot for experimental straight-ahead walking [35]. Similar advances will enable humanoid robots to employ our asymptotically stable motion planning framework.

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