Bringing the Compass-Gait Bipedal Walker to Three Dimensions

Robert D. Gregg and Mark W. Spong

***Abstract*— The planar compass-gait biped has been extensively studied in the dynamic walking community, motivated by the gravity-based pendular efficiencies of human walking. These results can be extended to three dimensions using controlled geometric reduction for open-chain robots, by which stable 3-D walking gaits are built from known sagittal-plane limit cycles. We apply this method to the standard and with-torso compass-gait (hipless) bipeds, showing straight-ahead walking gaits (i.e., stable 1-step periodic limit cycles) as well as h-step turning in full circles (i.e., stable h-periodic limit cycles). These constant-curvature maneuvers are composed of stable 1-periodic turning gaits modulo heading change, demonstrating two types of gaits for directional dynamic walking in three dimensions.**

I. Introduction

The first studies on dynamic bipedal walking in the robotics community concerned simple serial-chain models constrained to the sagittal plane (2-D space) to roughly approximate efficient human locomotion. This began with the two-link "compass-gait" biped in the pioneering work of [9]. McGeer discovered that this biped has stable "passive" limit cycles (i.e., uncontrolled walking gaits) down slopes between about 3° and 5°, the range of angles for which the potential energy introduced by gravity over each stride is matched by the energy dissipated at foot-ground impact. The behavior of these passive gaits with increasing slope angle was studied in [3], showing period-doubling bifurcations leading to chaos.

Actuated compass-gait models with knees and a torso were considered in [14], [15], generating stable planar walking gaits by zeroing hybrid-invariant virtual constraints (i.e., hybrid zero dynamics). Passive dynamics were directly exploited in [12], [13], using passivity-based control and controlled symmetries to map passive limit cycles down slopes to "pseudo-passive" limit cycles on arbitrary slopes with expanded basins of attraction. However, because stable passive limit cycles are rare in three dimensions, this method was primarily applied to various forms of the planar compass-gait biped. Such gaits were shown to be time-scalable for varied walking speeds in [7].

These well-studied passivity results were harnessed to build pseudo-passive 3-D walking gaits by exploiting inherent robot symmetries in [1], [2]. Ames et al. showed that controlled geometric reduction can decompose a spatial 3-D biped's dynamics into the sagittal plane-of-motion and a separate lean mode in the frontal/lateral plane. This was generalized to controlled reduction by stages in [5], [6], allowing application to completely 3-D bipeds with both yaw and lean modes. The resulting sagittal subsystem has the dynamics of an associated planar biped, from which stable full-order walking cycles are built about arbitrary headings.

This simplifies the search for full-order limit cycles and expands the class of 3-D robots that can achieve pseudo-passive dynamic walking. The above papers demonstrate this principle for hipped bipeds, showing basic steering capabilities, but offer no characterization of stability over turning maneuvers. Directional dynamic walking is also achieved for underactuated bipeds based on hybrid zero dynamics in [10]. These gaits are shown input-to-state stable for sufficiently small steering motions, but stability over paths with large total curvature is not considered.

This paper revisits controlled reduction to extend the planar compass-gait biped (both the standard two-link and with-torso three-link models) into three dimensions. This fully-actuated[1] "toy" biped captures the fundamental periodic motion involved in dynamic walking, allowing us to address the construction and stability of dynamic gaits for both straight-ahead and curved walking. In particular, we show that constant-curvature turning induces stable periodic limit cycles modulo heading change. This provides multiple types of walking gaits for directed locomotion in three dimensions.

II. Controlled Reduction

Classical geometric reduction is an analytical tool for decomposing a physical system, often modeled by a Lagrangian function, with symmetries that are invariant under the action of a Lie group on the configuration space. A few such forms of reduction are discussed in [8], such as Lie-Poisson, Euler-Poincaré, and Routh. In classical *Routhian reduction*, a Lagrangian L has configuration space $Q = \mathbb{G} \times S$ (usually an n-torus), where $\mathbb{G} = \mathbb{G}_1 \times \ldots \times \mathbb{G}_k$ is composed of *symmetry groups* and $S \cong Q \backslash \mathbb{G}$ is the *shape space*. Symmetries of L are characterized by *cyclic* variables $q_i \in \mathbb{G}_i$, such that

$$\frac{\partial L}{\partial q_i} = 0, \quad i \in \{1, k\}. \tag{1}$$

By dividing out the symmetry group $\mathbb{G}$, the full-order phase space[2] TQ projects onto the reduced-order phase space TS. Moreover, Equation (1) implies that each cyclic coordinate's generalized momentum is constant. When the dynamics evolve on level-sets of these conserved momentum quantities, the symmetries allow us to directly relate the behavior of the full-order system and the reduced system.

R. D. Gregg is with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 rgregg@illinois.edu

M. W. Spong is with the Department of Electrical Engineering, University of Texas at Dallas, Richardson, TX 75080 mspong@utdallas.edu

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[1] We consider a minimal-DOF hipless extension of this planar biped, which can only actuate out-of-plane motion with full actuation at the ankle.

[2] Tangent bundle TQ: space of configurations and their tangent velocities.

$$M(q_2^n) = \begin{pmatrix} m_{q_1}(q_2^n) & M_{q_1,q_2^n}(q_2^n) \\ M^T_{q_1,q_2^n}(q_2^n) & M_{q_2^n}(q_3^n) \end{pmatrix} = \begin{pmatrix} m_{q_1}(q_2^n) & — & M_{q_1,q_2^n}(q_2^n) & —— \\ | & \ddots & & \vdots \\ M^T_{q_1,q_2^n}(q_2^n) & & m_{q_{i-1}}(q_i^n) & M_{q_{i-1},q_i^n}(q_i^n) \\ | & \cdots & M^T_{q_{i-1},q_i^n}(q_i^n) & M_{q_i^n}(q_{i+1}^n) \end{pmatrix} \tag{2}$$

$$\begin{aligned} L_{\lambda_j^k}(q_j^n, \dot{q}_j^n) &= K_{\lambda_j^k}(q_j^n, \dot{q}_j^n) - V_{\lambda_j^k}(q_j^n) = \frac{1}{2}\dot{q}_j^{n^T} M_{\lambda_j^k}(q_{j+1}^n)\dot{q}_j^n - W_{\lambda_j^k}(q_j^n, \dot{q}_{j+1}^n) - V_{\lambda_j^k}(q_j^n) && \text{(3a)} \\ &= L_{\lambda_{j+1}^k}(q_{j+1}^n, \dot{q}_{j+1}^n) + \frac{1}{2} m_{q_j}(q_{j+1}^n)(\dot{q}_j)^2 + \dot{q}_j M_{q_j,q_{j+1}^n}(q_{j+1}^n)\dot{q}_{j+1}^n && \text{(3b)} \\ &\quad + \frac{1}{2}\dot{q}_{j+1}^{n^T} \frac{M^T_{q_j,q_{j+1}^n}(q_{j+1}^n) M_{q_j,q_{j+1}^n}(q_{j+1}^n)}{m_{q_j}(q_{j+1}^n)} \dot{q}_{j+1}^n - \frac{\lambda_j(q_j)}{m_{q_j}(q_{j+1}^n)} M_{q_j,q_{j+1}^n}(q_{j+1}^n)\dot{q}_{j+1}^n + \frac{1}{2}\frac{\lambda_j(q_j)^2}{m_{q_j}(q_{j+1}^n)} \end{aligned}$$

$$M_{\lambda_j^k}(q_{j+1}^n) = M_{q_j^n}(q_{j+1}^n) + \sum_{i=j}^{k} \begin{pmatrix} 0_{i\times i} & 0_{i\times(n-i)} \\ 0_{(n-i)\times i} & \frac{M^T_{q_i,q_{i+1}^n}(q_{i+1}^n) M_{q_i,q_{i+1}^n}(q_{i+1}^n)}{m_{q_i}(q_{i+1}^n)} \end{pmatrix} \tag{4}$$

$$W_{\lambda_j^k}(q_j^n, \dot{q}_{j+1}^n) = \sum_{i=j}^{k} \frac{\lambda_i(q_i)}{m_{q_i}(q_{i+1}^n)} M_{q_i,q_{i+1}^n}(q_{i+1}^n)\dot{q}_{i+1}^n, \qquad V_{\lambda_j^k}(q_j^n) = V_{\text{fct}}(q_{k+1}^n) - \frac{1}{2}\sum_{i=j}^{k} \frac{\lambda_i(q_i)^2}{m_{q_i}(q_{i+1}^n)} \tag{5}$$

In the context of walking, the divided coordinates correspond to unstable yaw and lean modes. Therefore, [1], [2] introduce a controlled form of geometric reduction, *functional Routhian reduction*, which breaks the symmetry of group $\mathbb{G}$ and provides momentum control of its coordinates. However, symmetry-breaking is imposed in a specific manner so that the group $\mathbb{G}$ of "almost-cyclic" variables can still be divided. In order to achieve multistage controlled reduction, [6] identifies extensive symmetries in serial-chain robot dynamics. This *recursively cyclic* property is exploited to show that any serial-chain robot can be controlled as a lower-dimensional subsystem. This is generalized in [5] by mapping branched chains to constrained serial chains.

We revisit this concept by presenting k-stage functional Routhian reduction for n-DOF serial chains, $1 \leq k < n$, later discussing reduction-based control for open-chain robots. We begin by describing a robot's typical Lagrangian dynamics.

Lagrangian Dynamics. A mechanical system with configuration space Q is described by elements $(q, \dot{q})$ of tangent bundle TQ and the Lagrangian function $L : TQ \to \mathbb{R}$, given in coordinates by

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q} - V(q),$$

where $K(q, \dot{q})$ is the kinetic energy, $V(q)$ is the potential energy, and $M(q)$ is the $n \times n$ symmetric, positive-definite inertia matrix. By the least action principle [8], L satisfies the n-dimensional controlled Euler-Lagrange (E-L) equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu. \tag{6}$$

This directly gives the dynamics for the controlled robot,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = Bu,$$

where $n \times n$-matrix $C(q, \dot{q})$ contains the Coriolis/centrifugal terms, $N(q) = \frac{\partial}{\partial q}V(q)$ is the vector of potential torques, $n \times n$-matrix B is assumed invertible for full actuation, and control input u is an n-vector of joint actuator torques.

These equations yield the dynamical control system (f, g):

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = f(q, \dot{q}) + g(q)u, \tag{7}$$

with vector field f and matrix g of control vector fields:

$$\begin{aligned} f(q, \dot{q}) &= \begin{pmatrix} \dot{q} \\ M(q)^{-1}\left(-C(q, \dot{q})\dot{q} - N(q)\right) \end{pmatrix} \\ g(q) &= \begin{pmatrix} 0_{n\times n} \\ M(q)^{-1}B \end{pmatrix}. \end{aligned}$$

If the Lagrangian has cyclic variables, as in (1), that are free from external forces (e.g., no actuation), we can decompose the dynamics with Routh reduction. However, in order to control these variables, we must describe a special class of Lagrangians that instead have "almost-cyclic" variables.

k-Almost-Cyclic Lagrangians. We start with a general n-dim. configuration space $Q = \mathbb{T}^k \times S$, where shape space $S \cong Q \backslash \mathbb{T}^k$ is constructed by $n-k$ copies of $\mathbb{R}$ and circle $\mathbb{S}^1$, and $\mathbb{T}^k = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is the group of "almost-cyclic" variables to be divided in stages of $\mathbb{S}^1$. We denote a configuration $q = (q_1, \ldots, q_n)^T = (q_1^{i^T}, q_{i+1}^{n^T})^T \in Q$ for $1 \leq i \leq n$, with i-dim. vector q_1^i containing coordinates $q_1, \ldots, q_i$ and $(n-i)$-dim. vector q_{i+1}^n containing $q_{i+1}, \ldots, q_n$ (clearly, if $i = n$ then $q_{n+1}^n = \emptyset$). In particular, for $i = k$ we have the vector of almost-cyclic variables $q_1^k \in \mathbb{T}^k$ and the vector of shape space variables $q_{k+1}^n \in S$. To begin, let inertia matrix M be defined from a class of *recursively cyclic* matrices, giving us the symmetries we need for reduction.

Definition 1: An $n \times n$-matrix M is *recursively cyclic* if each lower-right $(n-i+1) \times (n-i+1)$ submatrix is cyclic in $q_1, \ldots, q_i$ for $1 \leq i \leq n$, i.e., it has the form of (2) with base case $i = n$, where $M_{q_n^n}(q_{n+1}^n) = m_{q_n}$ is a scalar constant.

Remark 1: In our case, for $1 \leq i \leq n$, each $m_{q_i}(\cdot)$ is the scalar positive-definite self-induced inertia term of coordinate q_i, and $M_{q_i,q_{i+1}^n}(\cdot) \in \mathbb{R}^{i-1}$ is the row vector of off-diagonal inertial coupling terms between q_i and coordinates q_{i+1}^n.

Moreover, $M_{q_i^n}(\cdot) \in \mathbb{R}^{(n-i+1)\times(n-i+1)}$ is the symmetric positive-definite inertia submatrix of coordinates q_i^n.

A special class of shaped Lagrangians termed *almost-cyclic Lagrangians* is defined in [2], allowing one stage of controlled reduction to a subsystem characterized by a *functional Routhian* – the Lagrangian function of the lower-dim. system. In order to control k divided variables, each stage of reduction must project from an almost-cyclic Lagrangian (ACL) to another ACL for the next stage of reduction, until the final stage reaches the base functional Routhian. Therefore, we are interested in a generalized ACL.

Definition 2: A Lagrangian $L_{\lambda_1^k} : T\mathbb{T}^k \times TS \to \mathbb{R}$ is k-*almost-cyclic* if, in coordinates, it has the form

$$L_{\lambda_1^k}(q, \dot{q}) = K_{\lambda_1^k}(q, \dot{q}) - V_{\lambda_1^k}(q)$$

with expanded terms given in (3)-(5) for $j = 1$ and some arbitrary functions $\lambda_i : \mathbb{S}^1 \to \mathbb{R}$, $i \in \{1, k\}$.

Remark 2: The closed-form definition (3a) explicitly shows all the shaping terms necessary for k stages of controlled reduction, whereas the last three terms in recursive definition (3b) impose a single stage of controlled reduction to $(k-1)$-almost-cyclic Lagrangian $L_{\lambda_2^k}$.

Given k-almost-cyclic Lagrangian (k-ACL) $L_{\lambda_1^k}$, the n-dimensional fully-actuated E-L equations yield

$$M_{\lambda_1^k}(q_2^n)\ddot{q} + C_{\lambda_1^k}(q, \dot{q})\dot{q} + N_{\lambda_1^k}(q) = Bv.$$

Then, we have the control system on TQ associated with $L_{\lambda_1^k}$ as defined in (7): $(f_{\lambda_1^k}, g_{\lambda_1^k})$ with control input v.

This input can be decomposed into v_1^k, the vector containing the first k elements, and v_{k+1}^n, the $(n-k)$-vector containing elements $k+1, \dots, n$. Assuming subsystem input v_{k+1}^n is defined by a time-invariant feedback control law on TS, we incorporate this into the full-order k-ACL system by defining the new control system $(\hat{f}_{\lambda_1^k}, \hat{g}_{\lambda_1^k})$ with input v_1^k:

$$\begin{aligned} \hat{f}_{\lambda_1^k}(q, \dot{q}) &:= f_{\lambda_1^k}(q, \dot{q}) + g_{\lambda_1^k}(q) \begin{pmatrix} 0_{k\times 1} \\ v_{k+1}^n \end{pmatrix} \\ \hat{g}_{\lambda_1^k}(q) &:= g_{\lambda_1^k}(q) \begin{pmatrix} I_{k\times k} \\ 0_{(n-k)\times k} \end{pmatrix}. \end{aligned} \quad (8)$$

Here, vector field $\hat{f}_{\lambda_1^k}$ corresponds to the v_{k+1}^n-controlled E-L equations, which will be relevant later.

Reduced Subsystems. Starting with this k-ACL system, each reduction stage projects onto a lower-dimensional system while conserving a momentum quantity corresponding to the divided degree-of-freedom. These coordinates can thus be uniquely reconstructed by the functional momentum maps $J_i : T(Q\backslash\mathbb{T}^{i-1}) \to \mathbb{R}$, $i \in \{1, k\}$:

$$\begin{aligned} J_i(q_i^n, \dot{q}_i^n) &= \frac{\partial}{\partial \dot{q}_i} L_{\lambda_i^k}(q_i^n, \dot{q}_i^n) \\ &= M_{q_i, q_{i+1}^n}(q_{i+1}^n)\dot{q}_{i+1}^n + m_{q_i}(q_{i+1}^n)\dot{q}_i \\ &= \lambda_i(q_i). \end{aligned} \quad (9)$$

Here, $L_{\lambda_i^k}$ is the k-ACL for $i = 1$ or a lower-dim. ACL for $i \in \{2, k\}$. In classical Routh reduction, each J_i maps to a constant momentum quantity. However, the energy shaping terms in $L_{\lambda_1^k}$ break these conservative maps and force them equal to desirable functions $\lambda_i(q_i)$. Hence, we can control the momenta of the divided coordinates.

Each of these reduced subsystems is characterized by a generalized functional Routhian. For $j \in \{2, k\}$, the Routhian function corresponding to the $(j-1)^{\text{st}}$ stage of reduction is a $(k-j+1)$-ACL on the tangent bundle of reduced configuration space $Q\backslash\mathbb{T}^{j-1}$. This is the *stage-*$(j-1)$ *functional Routhian* $L_{\lambda_j^k} : T(Q\backslash\mathbb{T}^{j-1}) \to \mathbb{R}$, obtained through a partial Legendre transformation in q_{j-1} constrained to functional momentum map (9) for $i = j-1$:

$$\begin{aligned} L_{\lambda_j^k}(q_j^n, \dot{q}_j^n) &= L_{\lambda_{j-1}^k}(q_{j-1}^n, \dot{q}_{j-1}^n) - \lambda_{j-1}(q_{j-1})\dot{q}_{j-1}\Big|_{J_{j-1}} \\ &= K_{\lambda_j^k}(q_j^n, \dot{q}_j^n) - V_{\lambda_j^k}(q_j^n). \end{aligned}$$

We see that for $j \in \{2, k\}$, $L_{\lambda_j^k}$ has the form of (3).

The final stage of reduction, stage-k, is a functional Routhian $L_{\lambda_{k+1}^k} = L_{\text{fct}}$ with a traditional Lagrangian structure. This is similarly obtained from stage-$(k-1)$ functional Routhian $L_{\lambda_k^k}$. It follows that the k-reduced Lagrangian $L_{\text{fct}} : TS \to \mathbb{R}$ is given in coordinates by

$$L_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) = \frac{1}{2}\dot{q}_{k+1}^{n\,T} M_{q_{k+1}^n}(q_{k+1}^n)\dot{q}_{k+1}^n - V_{\text{fct}}(q_{k+1}^n)$$

with target potential energy V_{fct}.

This yields the control system on TS associated with L_{fct}: $(f_{\text{fct}}, g_{\text{fct}})$ with input v_{k+1}^n. From this, we define the vector field corresponding to the k-reduced, controlled dynamics:

$$\hat{f}_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) := f_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) + g_{\text{fct}}(q_{k+1}^n)v_{k+1}^n. \quad (10)$$

When $v_1^k = 0$ and the functional momentum quantities abide by (9), there exists a map between solutions of full-order vector field $\hat{f}_{\lambda_1^k}$ and reduced-order vector field $\hat{f}_{\text{fct}}$.

Theorem 1: Let $L_{\lambda_1^k}$ be a k-ACL with stage-k functional Routhian L_{fct}. Then, $\big(q_1^k(t), q_{k+1}^n(t), \dot{q}_1^k(t), \dot{q}_{k+1}^n(t)\big)$ is a solution to v_{k+1}^n-controlled vector field $\hat{f}_{\lambda_1^k}$ on $[t_0, t_f]$ with

$$J_j(q_j(t_0), \dot{q}_j(t_0)) = \lambda_j(q_j(t_0)), \quad \forall\, j \in \{1, k\},$$

if and only if $(q_{k+1}^n(t), \dot{q}_{k+1}^n(t))$ is a solution to controlled vector field $\hat{f}_{\text{fct}}$ on $[t_0, t_f]$ and $(q_j(t), \dot{q}_j(t))$ satisfies

$$J_j(q_j(t), \dot{q}_j(t)) = \lambda_j(q_j(t)), \quad \forall\, j \in \{1, k\},\ t \in [t_0, t_f].$$

Note that (9) can be solved for $\dot{q}_j$ to reconstruct each coordinate. We want to apply this form of controlled reduction, proven in [6], to general robots, but we still need to show that robots can attain the special k-almost-cyclic form.

Subrobot Theorem. We show in [6] that any serial chain has a recursively-cyclic inertia matrix, so k-ACL dynamics are achievable by energy shaping (i.e., *reduction-based* control). We further show in [5] that branched-chain robots have these symmetries through part of the chain. We now revisit these results for the encompassing class of open kinematic chains.

Definition 3: The *irreducible tree structure* of an n-DOF open chain is the minimal m-DOF tree substructure, $1 \leq m \leq n$, at the end of the chain such that the corresponding inertia submatrix is not recursively cyclic.

Lemma 1: For any fully-actuated n-DOF open-chain robot with an m-DOF irreducible tree structure, $1 \leq m < n$, and a potential energy that is cyclic in the first k coordinates, $1 \leq k \leq (n-m)$, there exists a feedback control law that shapes the system to the k-almost-cyclic form.

Theorem 2: Suppose an n-DOF open-chain robot has an m-DOF irreducible tree structure, $1 \leq m < n$, and a potential energy that is cyclic in the first k coordinates, $1 \leq k \leq (n-m)$. Then, the n-DOF robot is controlled-reducible down to its corresponding $(n-k)$-DOF subrobot.

Remark 3: Given the necessary symmetries in a robot's potential energy, we can impose a controlled reduction down to the m-DOF irreducible tree structure, which is trivially the last DOF for serial chains. Then, initial conditions satisfying (9) allow the shaped dynamics of an n-DOF robot to project onto the dynamics of the corresponding $(n-k)$-DOF subrobot. This subsystem is entirely decoupled from the first k coordinates and thus behaves and can be controlled as a typical $(n-k)$-DOF robot. Moreover, the first k DOF evolve in a controlled manner according to momentum constraint (9). These coordinates converge to set-points or periodic orbits based on the subsystem trajectories and our choice of functional momentum maps $\lambda(q_j) = -\alpha_j(q_j - \bar{q}_j)$, where α_j is a gain and $\bar{q}_j$ is a desired angle, for $j \in \{1, k\}$.

We now present our compass-gait biped models of interest.

III. Bipedal Walking Robots

A bipedal robot can be modeled as a hybrid system, which contains both continuous and discrete dynamics. We assume full actuation at the stance ankle of flat feet, which have instantaneous and perfectly plastic impact events. During the continuous swing phase, the contact between stance foot and ground is assumed flat without slipping. Note that some of these assumptions can be relaxed as in [10], [11]. We begin with some formalisms for hybrid systems from [2], [14].

Hybrid Systems. We consider simple hybrid systems with one continuous phase, i.e., "systems with impulse effects."

Definition 4: A *hybrid control system* has the form

$$\mathscr{HC} : \begin{cases} \dot{x} = f(x) + g(x)u & x \in D\backslash G \\ x^+ = \Delta(x^-) & x^- \in G \end{cases},$$

where $G \subset D$ is called the *guard* and $\Delta : G \to D$ is the *reset map*. In our context, state $x = (q^T, \dot{q}^T)^T$ is in domain $D \subseteq TQ$ and input u in control space $U \subset \mathbb{R}^n$. A *hybrid system* $\mathscr{H}$ has no explicit input u (e.g., a closed-loop system), and a *hybrid flow* is a solution to a hybrid system.

We also must define periodicity, since bipedal walking gaits correspond to periodic orbits of hybrid systems: straight-ahead gaits will be 1-step periodic and 360°-turning will be h-step periodic. Letting $x(t)$ be a hybrid flow of $\mathscr{H}$, it is *h-periodic* if $x(t) = x(t + \sum_{i=1}^{h} T_i)$, for all $t \geq 0$, where T_i is the fixed *time-to-impact* between the $(i-1)^{th}$ and i^{th} discrete events. Then, $x(t)$ is associated with its *h-periodic hybrid orbit* $\mathcal{O} = \{x(t) | t \geq 0\} \subset D$ from [14].

Stability of periodic hybrid orbits (i.e., limit cycles) is determined with the *Poincaré map* $P : G \to G$, which represents $\mathscr{H}$ as a discrete system between intersections with G (e.g., impact events). In particular, the h-composition of this map sends state $x_j \in G$ ahead h impact events by the discrete system $x_{j+h} = P^h(x_j)$. Thus, an h-periodic hybrid orbit $\mathcal{O}$ has an *h-fixed point* $x^* \in G \cap \mathcal{O}$ such that $x^* = P^h(x^*)$. We can numerically approximate the linearized map δP^h about x^*, by which we find the eigenvalues to determine local exponential stability (LES) of the discrete system [3].

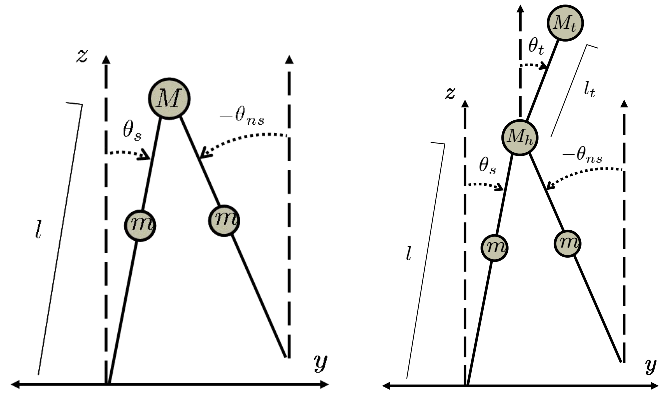

Fig. 1. The sagittal planes of the standard and torso compass-gait bipeds.

Biped Models. The models of application are the 4-DOF compass-gait and 5-DOF compass-gait-with-torso bipeds. These are 3-D extensions of the planar point-feet bipeds in Fig. 1, but the 3-D models do not have stable passive gaits down slopes. The first biped is a recursively cyclic serial chain, and torso bipeds are shown to be recursively cyclic down to the sagittal plane in [5]. Hence, Theorem 2 enables us to construct pseudo-passive 3-D gaits from sagittal-plane gaits. We begin by describing the hybrid control systems $\mathscr{HC}_{4\text{D}}$ and $\mathscr{HC}_{5\text{D}}$ associated with the standard and torso models, respectively, in terms of a general n-DOF biped.

In order to distinguish the sagittal-plane subsystem, we represent the configuration space of an n-DOF 3-D biped by $Q_{n\text{D}} = \mathbb{T}^2 \times \mathbb{T}^{n-2}$ with generalized coordinates $q = (\psi, \varphi, \theta^T)^T$, where ψ is the yaw (or heading), φ is the roll (or lean) from vertical, and θ is the vector of sagittal-plane (pitch) variables as in the associated 2-D model. In particular, the sagittal coordinates are $\theta = (\theta_\text{s}, \theta_\text{ns})^T$ for the 4-DOF model and $\theta = (\theta_\text{s}, \theta_\text{t}, \theta_\text{ns})^T$ for the 5-DOF model.

We derive the continuous dynamics with Lagrangian

$$L_{n\text{D}}(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T M_{n\text{D}}(q)\dot{q} - V_{n\text{D}}(q),$$

where $n \times n$ inertia matrix

$$M_{n\text{D}}(\varphi, \theta) = \begin{pmatrix} m_\psi(\varphi, \theta) & \text{---} & M_{\psi,\varphi,\theta}(\varphi, \theta) \\ | & m_\varphi(\theta) & M_{\varphi,\theta}(\theta) \\ M^T_{\psi,\varphi,\theta}(\varphi, \theta) & M^T_{\varphi,\theta}(\theta) & M_\theta(\theta) \end{pmatrix}$$

is recursively cyclic at least down to submatrix M_θ. The potential energy $V_{n\text{D}}(\varphi, \theta) = V_\theta(\theta)\cos(\varphi)$ contains the planar subsystem potential V_θ. Then, the controlled E-L equations (6) yield control system $(f_{n\text{D}}, g_{n\text{D}})$ with input u, subject to actuator saturation at torque constant $U^{max}_{n\text{D}}$.

We model walking on a flat surface by taking domain $D_{n\mathrm{D}} \subset TQ_{n\mathrm{D}}$ to be the set of states with nonnegative swing foot height. Impact events are triggered when this height is zero and decreasing, characterized by guard $G_{n\mathrm{D}} \subset D_{n\mathrm{D}}$ and impact map $\Delta_{n\mathrm{D}}$ (computed as in [6], [14]). We direct the reader to [4] for detailed term expressions, and now turn our attention to the reduction-based controller.

IV. Reduction-Based Control Law

The control law is designed to recursively break cyclic symmetries in the special almost-cyclic manner. The inner loop of the controller shapes our robot's energy to the 2-almost-cyclic form, and the nested outer loop plays two roles: implements passivity-based control on the 2-D-subsystem to construct known planar flat-ground gaits, and stabilizes to a surface defined by constraint (9) so that Theorem 1 holds. This builds upon the construction of the single-stage controller of [2]. We ignore actuator saturation during this derivation, but simulations will demonstrate robustness.

Lagrangian Shaping. The inner loop shapes $L_{n\mathrm{D}}$ into a 2-ACL for controlled reduction to the biped's planar subsystem. Given $q = (\psi, \varphi, \theta^T)^T$, the n-DOF potential $V_{n\mathrm{D}}$ is not cyclic in φ, so we impose a "controlled symmetry" with respect to this coordinate's rotation group $\mathbb{S}^1$. Therefore, we will incorporate potential shaping into the inner loop to replace $V_{n\mathrm{D}}$ with V_θ, the planar potential energy.

We begin with the 2-ACL (3a) for $j = 1$, $k = 2$:

$$L_{\lambda_1^2}(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T M_{\lambda_1^2}(q_2^n)\dot{q} - W_{\lambda_1^2}(q, \dot{q}_2^n) - V_{\lambda_1^2}(q),$$

where $M_{\lambda_1^2}$, $W_{\lambda_1^2}$, and $V_{\lambda_1^2}$ are defined by substituting inertia matrix $M_{n\mathrm{D}}$ for M and target potential V_θ for V_{fct} in (4)-(5). It follows directly that the stage-2 functional Routhian associated with $L_{\lambda_1^2}$ is the Lagrangian of the planar biped:

$$L_\theta(\theta, \dot{\theta}) = \tfrac{1}{2}\dot{\theta}^T M_\theta(\theta)\dot{\theta} + V_\theta(\theta),$$

yielding reduced control system (f_θ, g_θ) with input v_θ.

Given this target reduction, the feedback control law that shapes L into $L_{\lambda_1^2}$ is (11), where vector $v = (v_\psi, v_\varphi, v_\theta^T)^T$ contains the auxiliary control inputs to be defined. Finally, using momentum map functions $\lambda_1(\psi) = -\alpha_1(\psi - \bar{\psi})$ and $\lambda_2(\varphi) = -\alpha_2\varphi$, for $\alpha_1, \alpha_2 > 0$, we establish directional control to constant angle $\bar{\psi}$ for the yaw DOF and correction to vertical for the roll/lean DOF.

Inputting (11) into $(f_{n\mathrm{D}}, g_{n\mathrm{D}})$, we have shaped dynamics

$$M_{\lambda_1^2}(\varphi, \theta)\ddot{q} + C_{\lambda_1^2}(q, \dot{q})\dot{q} + N_{\lambda_1^2}(q) = B_{n\mathrm{D}}v$$

associated with $(f_{\lambda_1^2}, g_{\lambda_1^2})$ and input v to be defined next.

Passivity-Based Subsystem. Since controlled reduction can decouple a biped's sagittal subsystem, we can control it as a planar walker with well-known passivity-based techniques in v_θ. In particular, we employ slope-changing controlled symmetries, a method that imposes symmetries upon the system dynamics with respect to ground orientation. This will allow our bipeds to walk on flat ground given the existence of passive walking gaits down shallow slopes [13].

In order to harness known passive gaits, we use a subsystem controller that "rotates" the potential energy [13]:

$$v_\theta = B_\theta^{-1}\left(\tfrac{\partial}{\partial\theta}\left(V_\theta(\theta) - V_\theta(\theta + \beta)\right) + v_{pd}\right),$$

where $\beta = 0.052$ rad is the slope angle yielding the desired passive limit cycle. Note that the 5-DOF biped uprights its torso with PD control $v_{pd} = (0, -k_p(\theta_t + \beta) - k_d\dot{\theta}_t, 0)^T$.

This subsystem control law is incorporated into full-order shaped system $(f_{\lambda_1^2}, g_{\lambda_1^2})$ by defining the new control system $(\hat{f}_{\lambda_1^2}, \hat{g}_{\lambda_1^2})$ with input $v_1^2 = (v_\psi, v_\varphi)^T$ as in (8). Similarly, $\hat{f}_\theta$ is defined as in (10). In order to ensure the decoupling of $\hat{f}_\theta$, we now design v_1^2 to enforce constraint (9).

Zero Dynamics. The beneficial implications of Theorem 1 only hold from the set of states satisfying (9), so we must use control outside of this set to exploit the result. We extend the approach of [1] in using output linearization to force trajectories toward these conserved quantity constraints. We first define output functions measuring the error between actual and desired velocities according to (9):

$$h_i(q_i^n, \dot{q}_i^n) = \dot{q}_i - \tfrac{1}{m_{q_i}(q_{i+1}^n)}(\lambda_i(q_i) - M_{q_i, q_{i+1}^n}(q_{i+1}^n)\dot{q}_{i+1}^n)$$

for $i \in \{1, 2\}$. We use a control law that will zero these output functions in our MIMO nonlinear control system. In other words, we want to stabilize the "zero dynamics" surface

$$\mathcal{Z} = \left\{ \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \in TQ : h_i(q_i^n, \dot{q}_i^n) = 0, \ \ \forall i \in \{1, 2\} \right\}.$$

This law is proportional in nature, parameterized by gains ξ_i, and depends on Lie derivatives of h_i with respect to vector fields $(\hat{f}_{\lambda_1^2}, \hat{g}_{\lambda_1^2})$, but we leave the details to [6]. Note that $v_1^2|_{\mathcal{Z}} = 0$, so this law does not interfere with Theorem 1.

V. Results and Final Comments

We apply control law (11) under saturation to hybrid control system $\mathscr{HC}_{n\mathrm{D}}$, yielding closed-loop hybrid system $\mathscr{H}_{n\mathrm{D}}^{\mathrm{cl}}$. Our two bipeds are given physical and control parameters (12)-(13). We can use a restricted Poincaré map to prove the existence of LES straight-ahead gaits with arbitrary heading, as done in [1] for a spatial 3-D biped. This requires a lengthy proof involving hybrid invariance of surface $\mathcal{Z}$ along limit cycles, so we instead numerically verify LES with full-order Poincaré map P_{cl} as previously discussed [3].

Walking straight-ahead along any heading on flat ground (we set $\bar{\psi} = 0$ without loss of generality), $\mathscr{H}_{4\mathrm{D}}^{\mathrm{cl}}$ produces a hybrid periodic orbit $\mathcal{O}_{4\mathrm{D}}^{\mathrm{st}}$. The 1-fixed point $x_{4\mathrm{D}}^{*\mathrm{st}}$ is given in (14), from which we numerically calculate the eigenvalues of δP_{cl} to be within the unit circle. The same process confirms LES of $\mathcal{O}_{5\mathrm{D}}^{\mathrm{st}}$ with 1-fixed point $x_{5\mathrm{D}}^{*\mathrm{st}}$ of (15).

In order to command 360°-clockwise turning over h steps, we augment $\mathscr{H}_{n\mathrm{D}}^{\mathrm{cl}}$ with an event-based controller that increments desired yaw by $\Delta\bar{\psi} = 2\pi/h$ per step. Starting augmented system $\mathscr{H}_{4\mathrm{D}}^{\mathrm{tu}(h)}$ from $x_{4\mathrm{D}}^{*\mathrm{st}}$, we observe that hybrid flows converge to h-periodic $\mathcal{O}_{4\mathrm{D}}^{\mathrm{tu}(h)}$ for any $h \geq 13$, with 13-fixed point $x_{4\mathrm{D}}^{*\mathrm{tu}(13)}$ in (16). Similarly, $\mathscr{H}_{5\mathrm{D}}^{\mathrm{tu}(h)}$ yields $\mathcal{O}_{5\mathrm{D}}^{\mathrm{tu}(h)}$ for any $h \geq 14$, with 14-fixed point $x_{5\mathrm{D}}^{*\mathrm{tu}(14)}$ in (17). The sharpest 360° maneuvers are shown in Fig. 2. We find that

$$u_{n\mathrm{D}} := B_{n\mathrm{D}}^{-1}\left(C_{n\mathrm{D}}(q,\dot{q})\dot{q} + N_{n\mathrm{D}}(q) + M_{n\mathrm{D}}(q)M_{\lambda_1^2}(q)^{-1}\left(-C_{\lambda_1^2}(q,\dot{q})\dot{q} - N_{\lambda_1^2}(q) + B_{n\mathrm{D}}v\right)\right) \quad (11)$$

$$\text{Physical parameters} : M_t = 10 \text{ kg},\ l_t = 0.5 \text{ m},\ M_h = 10 \text{ kg},\ m = 5 \text{ kg},\ l = 1 \text{ m},\ U_{4\mathrm{D}}^{max} = 20 \text{ Nm},\ U_{5\mathrm{D}}^{max} = 30 \text{ Nm} \quad (12)$$

$$\text{Control parameters} : \alpha_1 = 4.5,\ \bar{\psi} = 0,\ \alpha_2 = 30,\ \xi_1 = 10,\ \xi_2 = 15,\ k_p = 700,\ k_d = 200,\ \beta = 0.052 \text{ rad} \quad (13)$$

$$x_{4\mathrm{D}}^{*\mathrm{st}} \approx (0, 0, -0.2704, 0.2704, 0, 0, -1.4896, -1.7986)^T \quad (14)$$

$$x_{5\mathrm{D}}^{*\mathrm{st}} \approx (0, 0, -0.2657, 0.0047, 0.2657, 0, 0, -1.3165, 0.0596, -1.5339)^T \quad (15)$$

$$x_{4\mathrm{D}}^{*\mathrm{tu}(13)} \approx (-0.0306, -0.0064, -0.2782, 0.2782, -0.0318, 0.0159, -1.5426, -2.1318)^T \quad (16)$$

$$x_{5\mathrm{D}}^{*\mathrm{tu}(14)} \approx (0.3231, 0.0492, -0.2717, -0.0030, 0.2717, 0.2895, -0.0443, -1.3510, 0.0865, -1.8180)^T \quad (17)$$

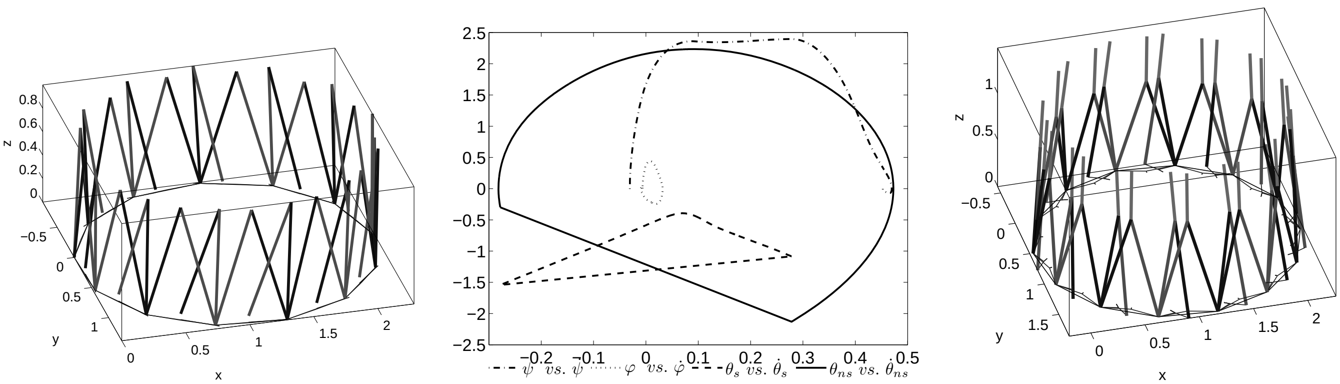

Fig. 2. 360°-CW turning maneuvers of 4-DOF biped over 13 steps (left) and 5-DOF biped over 14 steps (right). The former gait's phase portrait (center) shows planar slices of $\mathcal{O}_{4\mathrm{D}}^{\mathrm{tu}(13)}$ by plotting angular positions (modulo heading change) against angular velocities. This illustrates the orbit's periodicity in phase space, but note that this turning gait also has 1-periodic heading change $\Delta\psi = 0.4833$, step length 0.5493 m, and linear velocity 0.7412 m/s.

these fixed points are not only h-periodic LES by $\delta P^h_{\mathrm{tu}(h)}$, but also 1-periodic LES modulo heading change:

$$\mathrm{mod}_\psi\left(x_{n\mathrm{D}}^{*\mathrm{tu}(h)}, \Delta\psi\right) = P_{\mathrm{tu}(h)}\left(x_{n\mathrm{D}}^{*\mathrm{tu}(h)}\right).$$

In fact, constant-curvature steering with any sufficiently small $\Delta\bar{\psi}$ appears to induce a 1-step periodic gait, which has natural leaning into the turn (Fig. 2). Clockwise and CCW gaits are symmetric within the sagittal plane and otherwise antisymmetric. These gaits have perturbed sagittal-plane orbits compared to straight-ahead gaits, due to impact discontinuities in the conserved quantities while turning. This ultimately causes period-doubling instability, associated with oscillating step length/time, as periodic $\Delta\bar{\psi}$ is increased outside the observed stability range. This behavior resembles the passive gait bifurcations of the planar compass-gait biped when increasing slope angle [3], which merits further study.

We have extended passivity-based walking gaits of the planar compass-gait biped to 3-D, and this could similarly be applied to other results from the compass-gait literature (e.g., time-scaling [7], energy-tracking [12], zero dynamics [14]). Although this paper only considers hipless models, bipeds with hips have skew-symmetry between steps inducing 2-periodic limit cycles for straight and curved walking (mod $\Delta\psi$). We conclude by noting that these two types of gaits provide a minimal set of "motion primitives" for dynamic walking paths, which will be the subject of future work.

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